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A group theoretical calculation of the S matrix for the Dirac–Coulomb problem

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Abstract. Recently developed group theoretic techniques for calculating the S matrix are extended to the relativistic scattering regime. The Dirac-Coulomb problem is treated in detail. Phase shifts for an approximate Coulomb-Dirac Hamiltonian, which has $\overline{SO(3,1)}$ symmetry, are obtained for the first time, again using a group theoretical approach. The methods developed here may be useful in the construction of algebraic models for proton-nucleus scattering.

1. Introduction

Group theoretical techniques for scattering problems have proved to be useful in calculating the scattering matrix of quantum systems [1-4].

The work done so far, however, has been restricted to the non-relativistic Schrödinger system, and the question arises as to whether or not a similar method can be used in the group theoretical approach to relativistic scattering problems. We shall answer this question in this paper.

We first review the Lie-algebraic techniques and results in solving the non-relativistic Coulomb scattering [5, 6] and the formalism for the Dirac scattering problem [7]. Then, we discuss an approximate Dirac-Coulomb Hamiltonian possessing $\overline{SO(3,1)}$ symmetry [8]. Finally, we extend the technique to the Dirac-Coulomb scattering problem itself. Since the Dirac-Coulomb problem is an example where symmetry breaking occurs, the technique developed in calculating Dirac-Coulomb scattering may shed light on the generalisation of this technique to the cases where no appropriate dynamical symmetry exists.

2. Non-relativistic Coulomb scattering

It is well known [5] that the non-relativistic Coulomb Hamiltonian

$$H=\frac{1}{2M}\boldsymbol{p}^2+\frac{\alpha}{r} \tag{2.1}$$

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possesses two vector invariants: the angular momentum,

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \tag{2.2}$$

and the Lenz vector

$$\boldsymbol{A} = \frac{1}{2M}(\boldsymbol{p} \times \boldsymbol{L} - \boldsymbol{L} + \boldsymbol{p}) + \alpha \hat{\boldsymbol{r}} \tag{2.3}$$

where $\hat{\boldsymbol{r}} \equiv \boldsymbol{r}/r$. (Note that we have chosen a *positive* sign for the potential corresponding to a repulsive interaction.)

Since $\boldsymbol{L}$ and $\boldsymbol{A}$ commute with H we can renormalise the Lenz vector by introducing

$$\boldsymbol{K} \equiv (M/2H)^{1/2}\boldsymbol{A}. \tag{2.4}$$

We may then restrict H to a subspace where the Hamiltonian has a definite eigenvalue, i.e.

$$H \to \frac{1}{2M}k^2$$

so that

$$\boldsymbol{K} \to (M/k)\boldsymbol{A}. \tag{2.4a}$$

It can be shown that $\boldsymbol{L}$ and $\boldsymbol{K}$ generate the SO(3, 1) symmetry of the non-relativistic Coulomb scattering, i.e. they satisfy the following commutation relations

$$\begin{aligned} &[L_i, L_j] = \mathrm{i}\varepsilon_{ijk}L_k \\ &[L_i, K_j] = \mathrm{i}\varepsilon_{ijk}K_k \\ &[K_i, K_j] = -\mathrm{i}\varepsilon_{ijk}L_k. \end{aligned} \tag{2.5}$$

The SO(3, 1) representation generated by the realisation (2.2), (2.4) is characterised by a pair of numbers [9]

$$l_0 = 0 \qquad c = \mathrm{i}\alpha' \tag{2.6}$$

where $\alpha' = \alpha M/k$. They are related to the two SO(3, 1) invariants by

$$\begin{aligned} &\boldsymbol{L} \cdot \boldsymbol{K} = \mathrm{i}l_0 c = 0 \\ &\boldsymbol{L}^2 - \boldsymbol{K}^2 = l_0^2 + c^2 - 1 = -1 - \alpha'^2. \end{aligned} \tag{2.7}$$

The SO(3, 1) basis states $|\alpha' lm\rangle$ are given by

$$\begin{aligned} &(\boldsymbol{L}^2 - \boldsymbol{K}^2)|\alpha' lm\rangle = -(1 + \alpha'^2)|\alpha' lm\rangle \\ &\boldsymbol{L}^2|\alpha' lm\rangle = l(l+1)|\alpha' lm\rangle \\ &L_3|\alpha' lm\rangle = m|\alpha' lm\rangle. \end{aligned} \tag{2.8}$$

Let us now consider the group theoretic approach to scattering [11]. The Lie algebra describing the symmetry must be obeyed everywhere in $\mathbb{R}^3$, with exactly the same formal relations. The form of the generators may change (say in regions distant from the charge centre), but the algebra *per se* cannot anywhere.

Scattering is, however, a matter of asymptotics, with the initial and final states measured at infinity. This does not imply that the asymptotic symmetry is Euclidean—indeed for the Coulomb field it is well known that the motion never becomes interaction free.

What does change is the space over which the realisation is constructed. The space of scattering states is obtained by deleting the origin and computing the contraction limit (as the result of factoring out the radial motion) of all operators. For the problem at hand, this yields the 2-sphere at infinity (S^2) augmented by a two-element group $\mathbb{Z}_2$, the 'remnant' of the in- and out-going waves. Thus the factor space of scattering states is $S^2 \times \mathbb{Z}_2$, with $\mathbb{Z}_2$ generated by R (with $R^2 = E$). The operator R is the limit of the radial operator $(ik)^{-1}\partial/\partial r$.

The generators $\boldsymbol{L}$, equation (2.2), and $\boldsymbol{K}$, equation (2.4), acting on the factor space $S^2 \times \mathbb{Z}_2$, can be explicitly constructed by taking the asymptotic limit for large r. Since $\boldsymbol{L}$ acts on S^2—and commutes with R—we see that $\boldsymbol{L}$ is unchanged. The remaining operator $\boldsymbol{K}$, in the limit, takes the form:

$$\boldsymbol{K} \to \boldsymbol{K}' \equiv \tfrac{1}{2}(\hat{r} \times \boldsymbol{L} - \boldsymbol{L} \times \hat{r})R + \alpha' \hat{r}. \tag{2.9}$$

It is easily shown that the operators $\boldsymbol{L}$, $\boldsymbol{K}'$ and R (with $R^2 = E$) generate the group $SO(3,1) \times \mathbb{Z}_2$.

The ket vectors—denoted by the ket $\{|\alpha' lm, \varepsilon\rangle\}$—in the space of scattering states, $S^2 \times \mathbb{Z}_2$, are labelled by the eigenvalue equations:

$$\boldsymbol{L} \cdot \boldsymbol{K}' \to 0 \tag{2.10a}$$

$$\boldsymbol{L}^2 - (\boldsymbol{K}')^2 \to -1 - (\alpha')^2 \tag{2.10b}$$

$$\boldsymbol{L}^2 \to l(l+1) \tag{2.11a}$$

$$L_3 \to m \tag{2.11b}$$

which are inherited from (2.7) and (2.8), plus the additional label

$$R \to \varepsilon = \pm 1 \tag{2.12}$$

which denotes the in ($\varepsilon = -1$)/out ($\varepsilon = +1$) character of the limiting asymptotic states.

For the SO(3, 1) representation, there exist an l-raising operator $X_+(l)$ and an l-lowering operator $X_-(l)$, which are defined by [10]:

$$\begin{aligned} X_+(l) &= K_- L_+ + K_3 L_3 + (l+1)K_3 \\ X_-(l) &= K_+ L_- + K_3 L_3 + lK_3 \end{aligned} \tag{2.13}$$

with

$$K_\pm = K_1 \pm iK_2 \qquad L_\pm = L_1 \pm iL_2.$$

These raising/lowering operators exist for both realisations, equations (2.3) and (2.9), of the SO(3, 1) algebra.

For the standard realisation, equation (2.3), with the standard phasing and normalisation [12] of the ket vectors $|\alpha' lm\rangle$, it can be shown that

$$X_+(l)|\alpha' lm\rangle = \left(\frac{(2l+1)[(l+1)^2 - m^2][(l+1)^2 + \alpha'^2]}{2l+3}\right)^{1/2} (-i)|\alpha' l+1\, m\rangle. \tag{2.14}$$

It should be noted that this standard group theoretic matrix action used in (2.14) above, is not canonical, but requires arbitrary (but fixed) phase conventions [12]. Indeed the realisation of SO(3, 1) by equations (2.9) is easily seen not to be in the standard form. Only by an appropriate re-phasing of the scattering space kets, $|\alpha' lm; \varepsilon\rangle$, can the standard matrix action be restored. Physically this fact is obvious, since the required re-phasing is precisely the introduction of scattering phases for the asymptotic

states [11]. We conclude that the original kets $|\alpha' lm\rangle$ and the kets $|\alpha' lm; \varepsilon\rangle$ in the factor space of scattering states are related by the asymptotic equivalence relation:

$$|\alpha' lm\rangle \sim A_{lm}(k)|\alpha' lm; \varepsilon = -1\rangle + B_{lm}(k)|\alpha' lm; \varepsilon = +1\rangle. \tag{2.15a}$$

It can be shown by using O(3) m-raising/lowering operators on both sides of equation (2.15a) that $A_{lm}(k)$ and $B_{lm}(k)$ are independent of m. We drop the subscript m in the following discussion.

Let us now operate on (2.15a) with the l-raising operator $X_+(l)$. On the left-hand side, the action is standard and we get the matrix element of (2.14). On the right-hand side the action is non-standard, using (2.9), which we evaluate directly to be:

$$X'_+(l)|\alpha' lm; \varepsilon\rangle = -\left(\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}\right)^{1/2} (2l+1)[\varepsilon(l+1) + i\alpha']|\alpha' l+1 m, \varepsilon\rangle. \tag{2.15b}$$

These results (using (2.15a) again with $l \to l+1$) lead to the recursion relations:

$$-\left(\frac{(2l+1)[(l+1)^2+\alpha'^2]}{2l+3}\right)^{1/2} iA_{l+1} = \left(\frac{2l+1}{2l+3}\right)^{1/2} (l+1-i\alpha')A_l$$

$$-\left(\frac{(2l+1)[(l+1)^2+\alpha'^2]}{2l+3}\right)^{1/2} iB_{l+1} = -\left(\frac{2l+1}{2l+3}\right)^{1/2} (l+1+i\alpha')B_l$$

i.e.

$$\begin{aligned} A_{l+1} &= i\left(\frac{l+1-i\alpha'}{l+1+i\alpha'}\right)^{1/2} A_l \\ B_{l+1} &= -i\left(\frac{l+1+i\alpha'}{l+1-i\alpha'}\right)^{1/2} B_l. \end{aligned} \tag{2.16}$$

The recursion relation for the reflection amplitudes, which is defined as $R_l(k) = B_l(k)/A_l(k)$, is then

$$R_{l+1}(k) = -\frac{l+1+i\alpha'}{l+1-i\alpha'} R_l(k). \tag{2.17}$$

The solution to this recursion relation is

$$R_l(k) = (-1)^l \frac{\Gamma(l+1+i\alpha')}{\Gamma(l+1-i\alpha')} \Delta(k). \tag{2.18}$$

The S matrix elements are related to the reflection amplitudes by

$$S_l(k) = e^{il\pi} R_l(k). \tag{2.19}$$

Therefore, we have

$$S_l(k) = \frac{\Gamma(l+1+i\alpha')}{\Gamma(l+1-i\alpha')} \Delta(k) \tag{2.20}$$

where $\Delta(k)$ is independent of l and to be determined by the S-wave phase shift.

3. The Dirac equation and its scattering matrix

Before extending the algebraic approach to Dirac systems, we shall first introduce the notation to be used in the discussion.

In the Dirac problem, the wavefunction ψ describing the scattering has four components ψ_λ, which have the asymptotic form [7] (for non-Coulombic central potentials):

$$\psi_\lambda \sim a_\lambda \, e^{ikz} + r^{-1} \, e^{ikr} f_\lambda(\theta, \phi) \qquad (\lambda = 1, 2, 3, 4). \tag{3.1}$$

The corresponding differential cross section is

$$\frac{d\sigma}{d\Omega} = \sum_1^4 |f_\lambda(\theta, \phi)|^2 \left(\sum_1^4 |a_\lambda|^2 \right)^{-1}. \tag{3.2}$$

In the Dirac problem the quantities a_λ are not all independent. For the plane wave we have

$$\frac{|a_3|}{|a_1|} = \frac{|a_4|}{|a_2|} = \frac{k}{E+m}. \tag{3.3}$$

The same relation exists between the f_λ.

Therefore, (3.2) is reduced to

$$\frac{d\sigma}{d\Omega} = (|f_1(\theta, \phi)|^2 + |f_2(\theta, \phi)|^2)/(|a_1|^2 + |a_2|^2). \tag{3.4}$$

In the scattering problem the 'small' lower components do not provide further information.

For polarised beams, we have the following asymptotic forms:

$$\psi_1 \sim e^{ikz} + r^{-1} \, e^{ikr} f_1(\theta, \phi) \tag{3.5a}$$

$$\psi_2 \sim r^{-1} \, e^{ikr} g_1(\theta, \phi) \tag{3.5b}$$

and (with spins antiparallel to the direction of incidence)

$$\psi_1 \sim r^{-1} \, e^{ikr} g_2(\theta, \phi) \tag{3.6a}$$

$$\psi_2 \sim e^{ikz} + r^{-1} \, e^{ikr} f_2(\theta, \phi) \tag{3.6b}$$

where the scattering amplitudes $f_i(\theta, \phi)$, $g_i(\theta, \phi)$ $(i = 1, 2)$ are directly related to the elements of the S matrix.

For the partial wave expansion in a central field, we consider the solution of the Dirac equation in the following form:

$$(\psi)_\kappa = \begin{pmatrix} r^{-1} F_\kappa(r) \chi_\kappa^\mu \\ r^{-1} G_\kappa(r) \chi_{-\kappa}^\mu \end{pmatrix} \tag{3.7}$$

where the angular part χ_κ^μ is a two-component wavefunction defined by

$$\chi_\kappa^\mu = \sum_\tau \langle l\mu - \tau \tfrac{1}{2}\tau | j\mu \rangle \, Y_l^{\mu-\tau}(\theta, \phi) \chi_{1/2}^\tau. \tag{3.8}$$

Here, χ_κ^μ is the eigenfunction of $\mathcal{K} = -(\boldsymbol{\sigma} \cdot \boldsymbol{L} + 1)$, i.e.

$$\mathcal{K} \chi_\kappa^\mu = -(\boldsymbol{\sigma} \cdot \boldsymbol{L} + 1) \chi_\kappa^\mu = \kappa \chi_\kappa^\mu \tag{3.9}$$

which has a sharp angular momentum

$$j = |\kappa| - \tfrac{1}{2} \qquad j_3 = \mu \tag{3.10}$$

where

$$\boldsymbol{J} = \boldsymbol{L} + \tfrac{1}{2}\boldsymbol{\sigma}. \tag{3.11}$$

The important relation we are going to use in this paper is

$$(\boldsymbol{\sigma}\cdot\hat{r})\chi_{\kappa}^{\mu}=-\chi_{-\kappa}^{\mu} \tag{3.12}$$

and the relation that the orbital angular momentum is

$$l(\kappa)=|\kappa|+\tfrac{1}{2}[\mathrm{sgn}(\kappa)-1] \tag{3.13}$$

that is, corresponding to a fixed value of l, κ can take two values, l and $-l-1$. (Note that $\kappa=0$ is *excluded*: that is, $l=0$ corresponds uniquely to $\kappa=-1$.)

The solution (3.7) is the eigenfunction of Dirac's operator $K\equiv-\rho_3(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$, with eigenvalue κ. If we define the phase shifts of (3.7) to be δ_κ, that is

$$F_\kappa(r)\underset{r\to\infty}{\sim}A_\kappa\sin(kr-\tfrac{1}{2}\pi l(\kappa)-\eta\ln(2kr)+\delta_k) \tag{3.14}$$

(where we now are considering the (repulsive) Coulomb field), the relation between the scattering amplitudes and the S matrix of the Dirac system can be written as

$$f_1(\theta,\phi)=(2ik)^{-1}\sum_{l=0}^{\infty}\{(l+1)[\exp(2i\delta_l)-1]+l[\exp(2i\delta_{-l-1})-1]P_l(\cos\theta)\}$$

$$\equiv f(\theta) \tag{3.15a}$$

$$g_1(\theta,\phi)=(2ik)^{-1}\sum_{l=1}^{\infty}[-\exp(2i\delta_l)+\exp(2i\delta_{-l-1})]P_l^1(\cos\theta)\,e^{i\phi}\equiv g(\theta)\,e^{i\phi} \tag{3.15b}$$

and it can be proved that

$$\begin{aligned} f_2(\theta,\phi)&\equiv f(\theta)\\ g_2(\theta,\phi)&=-g(\theta)\,e^{-i\phi}. \end{aligned} \tag{3.16}$$

From (3.15) we can see that, similar to the non-relativistic scattering problem, an additional constant phase shift common to all $\delta_{\kappa=l}$ and $\delta_{\kappa=-l-1}$ will not contribute anything to the spin-flip and non-flip angular distributions at $\theta\neq0$, since

$$\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)=2\delta(\cos\theta-1).$$

This is one of the key points in the construction of algebraic models in a realistic scattering problem.

4. Scattering for an $\overline{\mathrm{SO}(3,1)}$ symmetric Dirac Hamiltonian

Before treating the Dirac-Coulomb problem, it is helpful to discuss first scattering from an approximate Dirac-Coulomb Hamiltonian which has $\overline{\mathrm{SO}(3,1)}$ symmetry [8]. The Hamiltonian, H_{sym}, for this relativistic $\overline{\mathrm{SO}(3,1)}$ symmetry is defined by subtracting the symmetry breaking term H_{fs} (where fs stands for fine structure)

$$H_{\mathrm{sym}}\equiv H_{\mathrm{Dirac}}-H_{\mathrm{fs}} \tag{4.1}$$

where

$$H_{\mathrm{Dirac}}\equiv\rho_1\boldsymbol{\sigma}\cdot\boldsymbol{p}+\rho_3 m+\alpha/r \tag{4.2}$$

and

$$H_{\mathrm{fs}}\equiv\rho_2(\boldsymbol{\sigma}\cdot\hat{r}/r)K\{[(1+(\alpha/K)^2]^{1/2}-1\}. \tag{4.3}$$

Consider the Dirac equation

$$H_{\text{sym}}\Psi = E\Psi. \tag{4.4}$$

In order to iterate equation (4.4) into a second-order differential equation we define the operator:

$$\begin{aligned} Q_+ &\equiv \rho_3(H_{\text{sym}} - E) \\ &= \mathrm{i}\rho_2\boldsymbol{\sigma}\cdot\boldsymbol{p} + m + \rho_3(\alpha/r - H_{\text{fs}} - E). \end{aligned} \tag{4.5}$$

Then we iterate the equation

$$Q_+\psi = 0 \tag{4.4'}$$

by multiplying on the left with Q_-, where

$$Q_- \equiv \mathrm{i}\rho_2\boldsymbol{\sigma}\cdot\boldsymbol{p} - m + \rho_3(\alpha/r - H_{\text{fs}} - E) \tag{4.6}$$

to obtain:

$$Q_+Q_-\psi = Q_-Q_+\psi = 0. \tag{4.4''}$$

It can be shown that equation (4.4'') has $\overline{\mathrm{SO}(3,1)}$ symmetry by introducing the similarity transformation S_1, where

$$S_1 \equiv \exp(\tfrac{1}{2}\rho_2(\boldsymbol{\sigma}\cdot\hat{r})\sinh^{-1}(\alpha/K)). \tag{4.7}$$

(This means that the symmetry is seen in the moving frame introduced by S_1.) The similarity transforms of $Q_\pm$ can be calculated to be:

$$Q'_\pm \equiv S_1 Q_\pm S_1^{-1} = \mathrm{i}\rho_2\boldsymbol{\sigma}\cdot\boldsymbol{p} \pm m - ES_1^2\rho_3 \tag{4.8}$$

and we have the relation:

$$Q'_-Q'_+\phi = Q'_+Q'_-\phi = \left(\nabla^2 + (E^2 - m^2) - \frac{2\alpha E}{r}\right)\phi = 0. \tag{4.9}$$

Accordingly, the solution of (4.4') is

$$\psi = Q_-S_1^{-1}\phi \tag{4.10}$$

if ϕ is a solution of (4.9).

Equation (4.9) has the same form as the non-relativistic Coulomb-Schrödinger equation:

$$\left(\boldsymbol{p}^2 + \frac{2k\alpha_r}{r}\right)\phi = k^2\phi \tag{4.9'}$$

with

$$\alpha_r = (\alpha/k)(k^2 + m^2)^{1/2}$$

and

$$k^2 = E^2 - m^2. \tag{4.11}$$

Notice that

$$\alpha_r \to \alpha m/k \qquad \text{as} \qquad k \to 0.$$

We restrict ourselves to the subspace where H_{sym} is definite, i.e. $E=|(k^2+m^2)^{1/2}|$. The six $\overline{\text{SO}(3,1)}$ generators are

$$\begin{aligned} \boldsymbol{L} &= \boldsymbol{r}\times\boldsymbol{p} \\ \boldsymbol{K} &= 1/k[\tfrac{1}{2}(\boldsymbol{p}\times\boldsymbol{L}-\boldsymbol{L}\times\boldsymbol{p})+\alpha E\hat{r}]. \end{aligned} \tag{4.12}$$

The states of the scattering factor space are the direct products of the states defined in the space $S^2\times\mathbb{Z}_2$ and the spin states; in the asymptotic algebra we have the operators $\hat{r}$, $\boldsymbol{L}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ available, along with the operator $R\to\varepsilon=\pm1$ in $\mathbb{Z}_2$.

The asymptotic forms of the $\overline{\text{SO}(3,1)}$ generators in (4.12) are now found to be:

$$\boldsymbol{L}\to\boldsymbol{L} \tag{4.13a}$$

$$\boldsymbol{K}\to\boldsymbol{K}'\equiv[\tfrac{1}{2}(\hat{r}\times\boldsymbol{L}-\boldsymbol{L}\times\hat{r})R+f(k)\hat{r}] \tag{4.13b}$$

where

$$f(k)\equiv\alpha_r\equiv\alpha\frac{(k^2+m^2)^{1/2}}{|k|} \tag{4.13c}$$

that is, the relativistic form of the Sommerfeld parameter. Repeating the discussion in § 2 for the solution of the second-order equation, we have the asymptotic equivalence relation:

$$|\phi\rangle\sim\begin{pmatrix} A_{l(\kappa)}|E\kappa\mu;\ \varepsilon=-1\rangle+B_{l(\kappa)}|E\kappa\mu;\ \varepsilon=+1\rangle \\ 0 \end{pmatrix} \tag{4.14}$$

where

$$l(\kappa=l)=l \tag{4.15}$$

and

$$l(\kappa=-l-1)=l. \tag{4.16}$$

From equations (4.13*a*, *b*), it follows group theoretically that, once again, $A_{l(\kappa)}$ and $B_{l(\kappa)}$ satisfy the recursion relation (2.16) (with α_r in the place of α').

The asymptotic equivalence relation for solutions to the first-order equation can be written as:

$$|\psi\rangle\sim Q_-^{\infty}(S_1^{-1})|\phi\rangle^{\sim}\qquad\text{as}\qquad r\to\infty \tag{4.17}$$

where $|\phi\rangle^{\sim}$ denotes the RHS of (4.14),

$$Q_-^{\infty}\equiv i\rho_2\boldsymbol{\sigma}\cdot\hat{r}kR-m-E\rho_3 \tag{4.18}$$

and

$$\begin{aligned} S_1^{-1} &= \exp[-\tfrac{1}{2}\rho_2(\boldsymbol{\sigma}\cdot\hat{r})\sinh^{-1}(\alpha/K)] \\ &= \left(\frac{[1+(\alpha/K)^2]^{1/2}+1}{2}\right)^{1/2} \\ &\quad -\operatorname{sgn}(K)\rho_2(\boldsymbol{\sigma}\cdot\hat{r})\left(\frac{[1+(\alpha/K)^2]^{1/2}-1}{2}\right)^{1/2}. \end{aligned} \tag{4.19}$$

We obtain

$$|\psi\rangle\sim\begin{pmatrix} C_\kappa^*(k)A_{l(\kappa)}|E\kappa\mu;\ \varepsilon=-1\rangle+C_\kappa(k)B_{l(\kappa)}|E\kappa\mu;\ \varepsilon=+1\rangle \\ k/(E+m)(C_\kappa^*(k)A_{l(\kappa)}|E-\kappa\mu;\ \varepsilon=-1\rangle-C_\kappa(k)B_{l(\kappa)}|E-\kappa\mu;\ \varepsilon=+1\rangle) \end{pmatrix} \tag{4.20}$$

where

$$C_\kappa(k) \equiv -(E+m)\left(\frac{[1+(\alpha/\kappa)^2]^{1/2}+1}{2}\right)^{1/2} - \mathrm{i}k\,\mathrm{sgn}(\kappa)\left(\frac{[1+(\alpha/\kappa)^2]^{1/2}-1}{2}\right)^{1/2}. \tag{4.21}$$

Notice that

$$C_{\kappa=-l-1}(k) = C^*_{\kappa=l+1}(k). \tag{4.22}$$

The phase shifts $\delta_{\kappa=l}$ and $\delta_{\kappa=-l-1}$ can be written as

$$\exp(2\mathrm{i}\delta_{\kappa=l}) = \exp(\mathrm{i}l(\kappa)\pi)\frac{B_{l(\kappa)}}{A_{l(\kappa)}}\exp(2\mathrm{i}\Delta_\kappa) = \frac{\Gamma(l+1+\mathrm{i}\alpha_r)}{\Gamma(l+1-\mathrm{i}\alpha_r)}\exp(2\mathrm{i}\Delta_\kappa)\,\mathrm{constant}(k) \tag{4.23}$$

and

$$\exp(2\mathrm{i}\delta_{\kappa=-l-1}) = \exp(\mathrm{i}l(\kappa)\pi)\frac{B_{l(\kappa)}}{A_{l(\kappa)}}\exp(2\mathrm{i}\Delta_\kappa) = \frac{\Gamma(l+1+\mathrm{i}\alpha_r)}{\Gamma(l+1-\mathrm{i}\alpha_r)}\exp(-2\mathrm{i}\Delta_\kappa)\,\mathrm{constant}(k) \tag{4.23'}$$

where

$$\Delta_\kappa = \tan^{-1}\left(\frac{k}{E+m}\,\frac{(\kappa^2+\alpha^2)^{1/2}-|\kappa|}{\alpha}\right). \tag{4.24}$$

It is interesting to notice that—for an *attractive* potential (that is, $\alpha_r \to -\alpha_r$)—the poles of the scattering matrix (4.23) provide us with the bound state spectrum. Since

$$l(\kappa)+1-\mathrm{i}\alpha_r = \text{negative integer} \equiv -(N-l(\kappa)-1) \tag{4.25}$$

we find that the principal quantum number N is:

$$\frac{\mathrm{i}\alpha E}{(E^2-m^2)^{1/2}} = N.$$

We have thus for the bound state energy E the result:

$$E = \frac{m}{[1+(\alpha/N)^2]^{1/2}} \qquad N = 1, 2, 3, \ldots. \tag{4.26}$$

(Note that these eigenenergies, and the wavefunctions in general, differ from the Dirac-Coulomb results in order $(\alpha Z)^2/|\kappa|$.)

5. Dirac–Coulomb scattering

The Dirac electron in a pure Coulomb field $\alpha Z/r$ (the Dirac-Coulomb problem) has the $\overline{\mathrm{SO}(3,1)}$ symmetry (of the symmetric Hamiltonian) broken by the fine-structure splitting, H_{fs}, to the symmetry $\overline{\mathrm{SO}(3)} \times \mathbb{Z}_2$. The $\mathbb{Z}_2$ symmetry here is generated by the Coulomb helicity operator [8]. Remarkably the operator form of this $\mathbb{Z}_2$ symmetry provides a complete definition of the Dirac-Coulomb radial function (in the 'most non-relativistic frame', cf [8]). The regular functions, satisfying the boundary condition

at $r=0$, can then be expanded asymptotically about the singular point at infinity, determining the relativistic Coulomb phase shifts. This latter step uses analysis for the confluent hypergeometric function and is neither group theoretic nor algebraic in nature.

To what extent can this procedure be made group theoretic?

To determine the answer let us consider again the symmetric Hamiltonian. In the quadratic Hamiltonian, (4.9), the Coulomb helicity operator is given by $\boldsymbol{\sigma}\cdot\boldsymbol{K}$, which (using (4.12)) becomes

$$\boldsymbol{\sigma}\cdot\boldsymbol{K}=(2k)^{-1}(\boldsymbol{\sigma}\cdot\boldsymbol{p}\times\boldsymbol{L}-\boldsymbol{\sigma}\cdot\boldsymbol{L}\times\boldsymbol{p})+\alpha_r\boldsymbol{\sigma}\cdot\hat{r}. \tag{5.1}$$

Using the identity:

$$\mathrm{i}(\boldsymbol{\sigma}\cdot\boldsymbol{p}\times\boldsymbol{L}-\boldsymbol{\sigma}\cdot\boldsymbol{L}\times\boldsymbol{p})=2\boldsymbol{\sigma}\cdot\boldsymbol{p}(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1) \tag{5.2}$$

we find that the Coulomb helicity operator takes the form:

$$\boldsymbol{\sigma}\cdot\boldsymbol{K}=\mathrm{i}k^{-1}\rho_3\boldsymbol{\sigma}\cdot\boldsymbol{p}K+\alpha_r\boldsymbol{\sigma}\cdot\hat{r} \tag{5.3}$$

where K is Dirac's operator.

By construction, $\boldsymbol{\sigma}\cdot\boldsymbol{K}$ commutes with the quadratic Hamiltonian, (4.9), and it is easily shown (from (5.3)) that K and $\boldsymbol{\sigma}\cdot\boldsymbol{K}$ anticommute. Since $(\boldsymbol{\sigma}\cdot\boldsymbol{K})^2$ obeys the relation $(\boldsymbol{\sigma}\cdot\boldsymbol{K})^2\to\kappa^2+\alpha_r^2$ on eigenkets, it follows that $\boldsymbol{\sigma}\cdot\boldsymbol{K}$ obeys the equation:

$$\boldsymbol{\sigma}\cdot\boldsymbol{K}|E,\kappa,\mu,\rho_3'\rangle=-(\kappa^2+\alpha_r^2)^{1/2}|E,-\kappa,\mu,\rho_3'\rangle \tag{5.4}$$

where $|E,\kappa,\mu,\rho_3'\rangle$ denotes the eigenkets of (4.9) having sharp energy E, $K\to\kappa$, $J_z\to\mu$, and $\rho_3\to\rho_3'=\pm1$.

Recall now that $j(\kappa)=|\kappa|-\frac{1}{2}$, that

$$l(\kappa)=|\kappa|+\left(\frac{\rho_3'\,\mathrm{sgn}(\kappa)-1}{2}\right)$$

and moreover that the radial eigenfunctions of (4.9) are functions of $l(\kappa)$. It follows that equation (5.4) acts as a raising/lowering operator in l for the eigenkets of the quadratic Hamiltonian (4.9).

Now let us consider the factor space of scattering states, $S^2\times\mathbb{Z}_2$, for this system in the energy sub-space E. We denote these eigenkets by: $|E,\kappa,\mu,\rho_3';\varepsilon\rangle^{\mathrm{scatt}}$ where E,κ,μ,ρ_3' are as above and ε denotes the eigenvalue of $R\to\varepsilon=\pm1$.

Using (4.13*a*, *b*), the operator $\boldsymbol{\sigma}\cdot\boldsymbol{K}$ takes on a very simple form:

$$\boldsymbol{\sigma}\cdot\boldsymbol{K}\to\boldsymbol{\sigma}\cdot\boldsymbol{K}^{\mathrm{scatt}}=\mathrm{i}\rho_3\boldsymbol{\sigma}\cdot\hat{r}RK+\alpha_r\boldsymbol{\sigma}\cdot\hat{r}. \tag{5.5}$$

(Note that the two terms in (5.5) anticommute.) It follows—just as for (5.4)—that $(\boldsymbol{\sigma}\cdot\boldsymbol{K})^{\mathrm{scatt}}$ acting on the scattering factor space eigenkets obeys the equation:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{K})^{\mathrm{scatt}}|E,\kappa,\mu,\rho_3';\varepsilon\rangle^{\mathrm{scatt}}=-(\mathrm{i}\rho_3'\kappa\varepsilon+\alpha_r)|E,-\kappa,\mu,\rho_3';\varepsilon\rangle^{\mathrm{scatt}}. \tag{5.6}$$

(The overall minus in (5.6), just as the minus in (5.4), stems from a phase definition in the action of $\boldsymbol{\sigma}\cdot\hat{r}$ on eigenkets of Dirac's operator K.)

We can now determine explicitly the phase shifts for the quadratic Hamiltonian using the asymptotic equivalence relation:

$$\begin{aligned}|E,\kappa,\mu,\rho_3'\rangle\sim{}&A_{-1}(E,l(\kappa),\rho_3')|E,\kappa,\mu,\rho_3';\varepsilon=-1\rangle^{\mathrm{scatt}}\\&+A_{+1}(E,l(\kappa),\rho_3')|E,\kappa,\mu,\rho_3';\varepsilon=+1\rangle^{\mathrm{scatt}}.\end{aligned} \tag{5.7}$$

(The constants $A_{\pm1}$ are independent of μ, as can be shown using raising/lowering operators.)

Taking, for definiteness, $\kappa = -l$ and $\rho_3' = +1$ and operating with $\boldsymbol{\sigma}\cdot\boldsymbol{K}(\boldsymbol{\sigma}\cdot\boldsymbol{K}^{\text{scatt}})$ on the two sides of (5.7), we obtain the general two-term recursion relation:

$$(l^2+\alpha_r^2)^{1/2}A_\epsilon(l) = [\mathrm{i}(-l)\varepsilon + \alpha_r]A_\epsilon(l-1) \qquad \text{for } \rho_3' = +1 \tag{5.8}$$

with the solution:

$$A_\epsilon(l) = (-\mathrm{i}\varepsilon)^l\left(\frac{\Gamma(l+1+\mathrm{i}\varepsilon\alpha_r)}{\Gamma(l+1-\mathrm{i}\varepsilon\alpha_r)}\right)^{1/2} \times (\text{a constant independent of } l). \tag{5.9}$$

We find the scattering phase shift:

$$\frac{A_{+1}}{A_{-1}} = (-1)^l\left(\frac{\Gamma(l+1+\mathrm{i}\alpha_r)}{\Gamma(l+1-\mathrm{i}\alpha_r)}\right) \times (\text{a constant independent of } l). \tag{5.10}$$

It follows that the scattering matrix has the explicit form:

$$S_\kappa = \exp(2\mathrm{i}\delta_\kappa)\Delta(k) \tag{5.11}$$

where $\Delta(k)$ is the constant independent of l, and $\delta_\kappa \equiv \arg(\Gamma(l(\kappa)+1+\mathrm{i}\alpha_r))$.

This procedure provides an alternative, purely group theoretic, derivation of the $\overline{\mathrm{SO}(3,1)}$ scattering for the second-order Hamiltonian (4.9), with the same results as obtained earlier in § 4. The advantage of the present method is that it generalises to the Dirac–Coulomb problem.

To carry out this generalisation we must now develop the analogue to $l(\kappa)$ for the second-order Dirac–Coulomb Hamiltonian. Let us define the first-order equation, analogous to (4.4′), by:

$$Q_+'\psi = 0 \tag{5.12}$$

where

$$Q_\pm' \equiv \rho_3(H_{\text{Dirac}} - E - \rho_3 m) \pm m \tag{5.13}$$

and

$$[Q_+', Q_-'] = 0. \tag{5.14}$$

Then the second-order equation has the form:

$$Q_+'Q_-'\psi = Q_-'Q_+'\psi = 0. \tag{5.15}$$

Expressed in spherical coordinates this equation, (5.15), is explicitly:

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{\Gamma(\Gamma+1)}{r^2} - \frac{2k\alpha_r}{r} + k^2\right)\phi = 0 \tag{5.16}$$

where the operator Γ is given by:

$$\Gamma = \rho_3 K - \mathrm{i}\alpha\rho_1\boldsymbol{\sigma}\cdot\hat{r}. \tag{5.17}$$

To put (5.16) in a more understandable form let us diagonalise Γ using the transformation:

$$S \equiv \exp[\tfrac{1}{2}\rho_2\boldsymbol{\sigma}\cdot\hat{r}\tanh^{-1}(\alpha/K)] \tag{5.18}$$

to obtain:

$$\Gamma \to \tilde{\Gamma} = S\Gamma S^{-1} = \rho_3 K|(1-(\alpha/K)^2)^{1/2}|. \tag{5.19}$$

Thus $\tilde{\Gamma}$ has the eigenvalues:

$$\tilde{\Gamma} \to \gamma = \pm|(\kappa^2 - \alpha^2)^{1/2}|. \tag{5.20}$$

The transformation S applied to (5.16) yields the result:

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(\gamma)(l(\gamma)+1)}{r^2} - \frac{2k\alpha_r}{r} + k^2\right) S\phi = 0 \tag{5.21}$$

which is readily interpretable: the radial Dirac-Coulomb eigenfunctions in the frame S have exactly the non-relativistic Coulomb functional form but with relativistic parameters: $k^2 = E^2 - m^2$, $\alpha_r = \alpha E/k$ and, most interestingly, irrational angular momenta: $l(\gamma) = |\gamma| + \frac{1}{2}(\mathrm{sgn}(\gamma) - 1)$. (Note that $\mathrm{sgn}(\gamma) = \mathrm{sgn}(\kappa)$.) We may interpret $l(\gamma)$ as the effective orbital angular momentum, defined by $L^2_{\mathrm{eff}} = \Gamma(\Gamma+1)$ when brought to diagonal form.

In the frame S it is not difficult to give an explicit operator expression for the Coulomb helicity operator:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{K})_{\text{Dirac-Coulomb}} = k^{-1}\boldsymbol{\sigma}\cdot\hat{r}\left(\mathrm{i}\hat{r}\cdot\boldsymbol{p} + \frac{1+\tilde{\Gamma}}{r}\right)\tilde{\Gamma} + \alpha_r\boldsymbol{\sigma}\cdot\hat{r} \equiv (\boldsymbol{\sigma}\cdot\boldsymbol{K})_{\text{D-C}}. \tag{5.22}$$

This expression for the Coulomb helicity operator anticommutes with K and commutes with the second-order (iterated) Dirac-Coulomb Hamiltonian. It is easily shown that the square of the operator in (5.22) is:

$$[(\boldsymbol{\sigma}\cdot\boldsymbol{K})_{\text{D-C}}]^2 = \gamma^2 + \alpha_r^2 \tag{5.23}$$

and that (in analogy to (5.4)) it obeys the equation:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{K})_{\text{D-C}}|E, \kappa, \mu, \rho_3'\rangle = -(\gamma^2+\alpha_r^2)^{1/2}|E, -\kappa, \mu, \rho_3'\rangle \tag{5.24}$$

where $|E, \kappa, \mu, \rho_3'\rangle$ denotes the eigenkets of (5.15).

These results show that the operator $(\boldsymbol{\sigma}\cdot\boldsymbol{K})_{\text{D-C}}$ is (when normalised) the generator of the symmetry group $\mathbb{Z}_2$ for the second-order Dirac-Coulomb Hamiltonian.

We can now determine the phase shifts for this (second-order) Hamiltonian, since (exactly as for the previous case of the symmetric Hamiltonian) equation (5.24) shows that $(\boldsymbol{\sigma}\cdot\boldsymbol{K})_{\text{D-C}}$ is a raising/lowering operator for the radial eigenfunctions, which now changes $l(\gamma)$ to $l(\gamma)\pm 1$.

Similarly, when we go to the factor space of scattering states, $S^2\times\mathbb{Z}_2$, (in the energy sub-space E) we have the eigenkets $|E, \kappa, \mu, \rho_3'; \varepsilon\rangle^{\text{scatt}}$ where ε denotes the eigenvalue of $R \to \varepsilon = \pm 1$. In this space, the operator $(\boldsymbol{\sigma}\cdot\boldsymbol{K})^{\text{scatt}}_{\text{D-C}}$ takes on a simple form:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{K})_{\text{D-C}} \to (\boldsymbol{\sigma}\cdot\boldsymbol{K})^{\text{scatt}}_{\text{D-C}} = \mathrm{i}\boldsymbol{\sigma}\cdot\hat{r}R\tilde{\Gamma} + \alpha_r\boldsymbol{\sigma}\cdot\hat{r}. \tag{5.25}$$

We use now the asymptotic equivalence relation:

$$|E, \kappa, \mu, \rho_3'\rangle \sim A_{-1}(E, l(\gamma), \rho_3')|E, \kappa, \mu, \rho_3'; \varepsilon = -1\rangle^{\text{scatt}} + A_{+1}(E, l(\gamma), \rho_3')|E, \kappa, \mu, \rho_3'; \varepsilon = +1\rangle^{\text{scatt}} \tag{5.26}$$

(where, as before, the A_ε are independent of μ).

Taking, for definiteness, as before $\kappa = -l$ and $\rho_3' = +1$, and operating on both sides of (5.26)—with $(\boldsymbol{\sigma}\cdot\boldsymbol{K})_{\text{D-C}}$ (on the LHS) and $(\boldsymbol{\sigma}\cdot\boldsymbol{K})^{\text{scatt}}_{\text{D-C}}$ (on the RHS)—we obtain the general two-term recursion relation:

$$(\gamma^2+\alpha_r^2)^{1/2}A_\varepsilon(l(\gamma) = |\gamma|) = [\mathrm{i}\varepsilon(-|\gamma|) + \alpha_r]A_\varepsilon(l(\gamma) = |\gamma| - 1). \tag{5.27}$$

This recursion relation is exactly of the same form as the recursion relation (5.8) except that $l(\gamma)=\gamma$ replaces l. If we were allowed to replace l by $l(\gamma)=\gamma$ in the solution (5.9), then we would get (for $\rho_3'=+1$) the scattering phase shift:

$$\frac{A_{+1}}{A_{-1}}=(-1)^{\gamma}\left(\frac{\Gamma(\gamma+1+i\alpha_r)}{\Gamma(\gamma+1-i\alpha_r)}\right)\times(\text{a constant independent of }\gamma). \tag{5.28}$$

It would then follow that the scattering matrix (for the second-order Dirac-Coulomb Hamiltonian) is given by:

$$S_\kappa=(-1)^{l-\gamma}\exp(2i\delta_\gamma)\Delta(k) \tag{5.29}$$

where

$$\gamma=|(\kappa^2-\alpha^2)^{1/2}| \qquad l=l(\kappa)=\text{integer} \qquad \delta_\gamma\equiv\arg(\Gamma(\gamma+1+i\alpha_r))$$

with $\Delta(k)$ a constant independent of all parameters except energy.

The scattering matrix given by (5.29) is correct, and agrees with the scattering matrix obtained using analysis on the regular solutions to (5.21), that is to say, the standard analysis of the confluent hypergeometric functions.

It is not clear that one can justify the solution (5.28), at least not in any strictly group theoretic way, without appeal to analysis. The use of the recursion relation, (5.27), does however fully justify the ratio of gamma functions; the problem only concerns the phase $(-1)^{\gamma}$. Since the recursion relation proceeds by integer steps, only $(-1)^{\text{integer}}$ can occur, with the solution (5.28) having an undetermined constant given by the initial value $\gamma-N$ (with $N=$ the number of steps).

For the symmetric Hamiltonian $l(\kappa)$ is an integer, so that $l=0$ is the standard initial value. For the Dirac-Coulomb case, each value of γ (since irrational) is a special case ($l(\gamma)=|\gamma|$ or $|\gamma|-1$) and, in fact, the relative phase shift between the different $|\gamma|$ is not determined (group theoretically). That is to say, there exists no common initial starting point for the recursion, valid for all the $|\gamma|$.

We conclude that, in the strict sense, a purely group theoretic determination of scattering phases of the (second-order) Dirac-Coulomb Hamiltonian is not possible. (This is probably not as surprising as it may appear, since already in [1] an explicitly energy-dependent phase (not obtainable by group theory) occurred in the complete answer.)

If we broaden the allowed methods to go beyond group theory (Lie-algebraic methods) and include general algebraic techniques (but not the methods of analysis) then we can justify the solution (5.28) as the algebraic replacement of $l(\kappa)$ by $l(\gamma)$ everywhere. (The phase $(-1)^l$ in (5.29) can be understood as having a different origin: the Jacobi-Anger identity.)

By contrast, the scattering matrix for the Dirac-Coulomb problem itself (that is, the first-order equation) follows precisely from the same method as used in § 4 for the (first-order) symmetric Hamiltonian.

The solutions of the first-order equation are given by:

$$|E,\kappa,\mu\rangle=Q_-'S^{-1}|E,\kappa,\mu,\rho_3'\rangle \tag{5.30}$$

where $|E,\kappa,\mu,\rho_3'\rangle$ is a solution of (5.15). The asymptotic solutions have the form:

$$|E,\kappa,\mu\rangle\sim Q_-^{\infty}S^{-1}|E,\kappa,\mu,\rho_3'\rangle^{\infty} \tag{5.31}$$

where $|E,\kappa,\mu,\rho_3'\rangle^{\infty}$ denotes the RHS of (5.26).

(Note that the asymptotic limit of Q'_- is the same as the asymptotic limit of Q_-, namely Q_-^∞ as given in (4.18).)

The operator S^{-1} is explicitly:

$$S^{-1}=\left(\frac{[1-(\alpha/K)^2]^{-1/2}+1}{2}\right)^{1/2} -\rho_2(\boldsymbol{\sigma}\cdot\hat{r})\left(\frac{[1-(\alpha/K)^2]^{-1/2}-1}{2}\right)^{1/2}\operatorname{sgn}(\alpha/K). \tag{5.32}$$

Notice that the form of Q_-^∞ is independent of the potential so long as $V(r)\to 0$, as $r\to\infty$. Moreover, S^{-1} is only determined by the diagonalisation of the effective orbital angular momentum. (These remarks could be useful in the construction of scattering models for the Dirac scattering and we will return to this in the conclusions.)

It remains to detail the actual solution in the original representation where we have the Dirac-Coulomb equation in the form given by (4.2). Writing (5.31) more explicitly we have

$$|E,\kappa,\mu\rangle\sim Q_-^\infty S^{-1}|E,\kappa,\mu,\rho_3'\rangle^\infty =\begin{pmatrix} a_\kappa^*(k)A_{-1}|E,\kappa,\mu,\rho_3';\ \varepsilon=-1\rangle+a_\kappa(k)A_{+1}|E,\kappa,\mu,\rho_3';\ \varepsilon=+1\rangle \\ k/(E+m)(a_\kappa^*(k)A_{-1}|E,-\kappa,\mu,\rho_3';\ \varepsilon=-1\rangle-a_\kappa(k)A_{+1}|E,-\kappa,\mu,\rho_3';\ \varepsilon=+1\rangle)\end{pmatrix} \tag{5.33}$$

where

$$a_\kappa(k)=-(E+m)\left(\frac{\kappa+\gamma}{2\gamma}\right)^{1/2}-ik\operatorname{sgn}(\alpha/\kappa)\left(\frac{\kappa-\gamma}{2\gamma}\right)^{1/2}. \tag{5.34}$$

Notice that

$$a_{\kappa=-l-1}(k)=a^*_{\kappa=l+1}(k). \tag{5.35}$$

The phase shifts $\eta_{\kappa=l}$ and $\eta_{\kappa=-l-1}$ are given by

(for $l\geqslant 1$)

$$\exp(2i\eta_{\kappa=l})=S_\kappa\exp(2i\phi_\kappa) =\exp\{i[l-(l^2-\alpha^2)^{1/2}]\pi\}\frac{\Gamma[(l^2-\alpha^2)^{1/2}+1+i\alpha_r]}{\Gamma[(l^2-\alpha^2)^{1/2}+1-i\alpha_r]}\exp(2i\phi_\kappa)\Delta(k) \tag{5.36}$$

and (for $l\geqslant 0$)

$$\exp(2i\eta_{\kappa=-l-1})=\exp[\![i\pi\{l+1-[(l+1)^2-\alpha^2]^{1/2}\}]\!] \times\frac{\Gamma\{[(l+1)^2-\alpha^2]^{1/2}+i\alpha_r\}}{\Gamma\{[(l+1)^2-\alpha^2]^{1/2}-i\alpha_r\}}\exp(-2i\phi_\kappa)\Delta(k) \tag{5.37}$$

where

$$\phi_\kappa=\tan^{-1}\left(\frac{k}{E+m}\frac{|\kappa|-(\kappa^2-\alpha^2)^{1/2}}{\alpha}\right). \tag{5.38}$$

It can be shown that $\exp(2i\phi_\kappa)$ with ϕ_κ given by (5.38) is identical to the standard expression given in [7], i.e.

$$\exp(2i\phi_l)=\frac{l-i\alpha_r(m/E)}{(l^2-\alpha^2)^{1/2}-i\alpha_r}. \tag{5.38'}$$

A closed form for the spin non-flip and flip elements (3.15) of the S matrix cannot be obtained. From these results, it is clear that the non-relativistic limit corresponds to $(l^2-\alpha^2)^{1/2}\to l$ and $\alpha_r\to\alpha'$, and that the symmetric Coulomb problem corresponds to $l\to(l^2+\alpha^2)^{1/2}$ only.

From the poles of the S matrix, (5.36) and (5.37), we can, for an *attractive* potential (that is, $\alpha_r\to-\alpha_r$), also derive the energy spectrum of the Dirac–Coulomb bound states. Since

$$l(\gamma)+1-\mathrm{i}\alpha_r=\text{negative integer}\equiv-(N-l(\kappa)-1)$$

i.e.

$$\mathrm{i}\alpha_r=N+|\gamma|-|\kappa| \tag{5.39}$$

we have

$$\alpha_r^2=\frac{\alpha^2E^2}{E^2-m^2}=-(N+|\gamma|-|\kappa|)^2 \tag{5.40}$$

or

$$E=m\left(1+\frac{\alpha^2}{(N+|\gamma|-|\kappa|)^2}\right)^{-1/2} \tag{5.41}$$

where

$$|\kappa|=j+\tfrac{1}{2}\qquad|\gamma|=(\kappa^2-\alpha^2)^{1/2}.$$

6. Conclusions

The symmetric (relativistic) Coulomb problem shows us that the Lie-algebraic procedures for the non-relativistic Schrödinger problem can indeed be extended in all details to the relativistic domain. Once we iterate the Dirac equation to get a second-order differential equation we can use similar techniques but we must also use more general algebraic methods. Since the algebraic procedure developed in the non-relativistic problem can be divorced from the specific differential realisation, so also can this procedure for the relativistic problem. The additional point is that we have to construct an operator to restore the solutions to the first-order equation. This could be done since $Q^{\infty}_{\pm}$ was independent of the specific potential form. All we have to do is to construct the effective angular momentum operator (the analogue to $\Gamma(\Gamma+1)$) and diagonalise it to find the diagonalising transformation S.

In a proton–nucleus collision, it is possible to set up a relativistic algebraic model by constructing an effective orbital angular momentum operator $\boldsymbol{L}^2_{\text{eff}}$, as suggested by the Dirac–Coulomb problem, to describe the modified Coulomb scattering. Here the modified Coulomb problem means that the short-range behaviour is modified and the long-range interaction is still of the nature of Coulomb scattering. We hope to report our progress along this line in the near future.

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